Lesson 3-7 Composition of Functions

Vocabulary

composite
function composition

BIG IDEA Following one process by another process creates a composite that itself can be viewed as a single process.

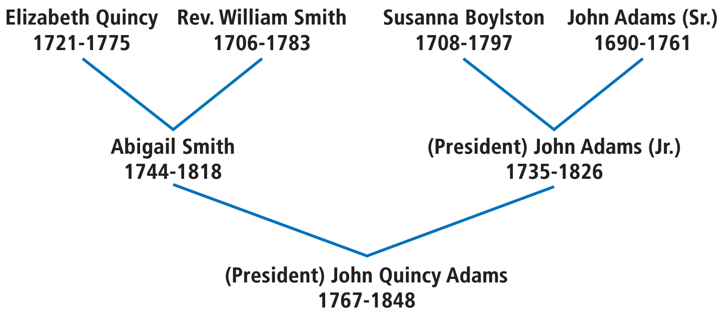
Mental Math

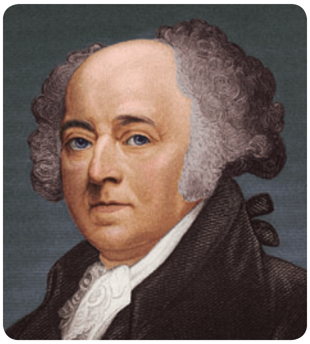
Start with a number *n*.

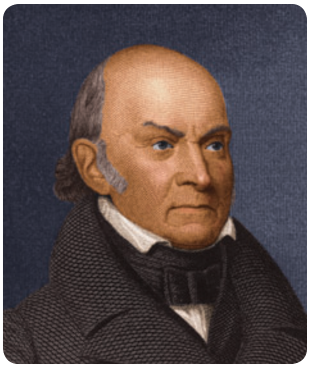
a. Subtract 4 from it. Then multiply the difference by 3. What number results?

b. Multiply it by 3. Then subtract 4 from the product. What number results?

The presidents of the United States from 1989–1992 and 2001–2008 were George H.W. Bush and his son. This was not the first time that a father and son had been presidents. The 2nd and 6th presidents, John Adams and John Quincy Adams, were also father and son. Here is part of John Quincy Adams's family tree. It shows that John Adams and Abigail Smith are John Quincy Adams's parents. Because President John Adams had the same name as his father, we call the father "Sr." and the son "Jr." here even though they never used those names.




John Adams


John Quincy Adams

Functions can be used to describe the relationships between members of this tree. For example, suppose m is the function defined by

$$m(x) \text{ is the mother of } x,$$

and f is the function defined by

$$f(x) \text{ is the father of } x.$$

Then m(John Quincy Adams) = Abigail Smith, m(Abigail Smith) = Elizabeth Quincy, f(John Quincy Adams) = John Adams Jr., and so on.

Functions can be combined so that the value of one function becomes the argument of another. For example, since

$$m(\text{John Quincy Adams}) = \text{Abigail Smith},$$

and

$$f(\text{Abigail Smith}) = \text{Rev. William Smith},$$

this combination can be written

$$f(m(\text{John Quincy Adams})) = f(\text{Abigail Smith}) = \text{Rev. William Smith}.$$

In words, this equation says that the father of the mother of John Quincy Adams is Rev. William Smith.

We say that m and f have been *composed* to make a new function, which we could call the "maternal grandfather" function. We denote this function by the symbol $f \circ m$, read as "the *composite* of f with m."

Definition of Composite Function

Suppose f and g are functions. The **composite** of g with f, written $\mathbf{g \circ f}$, is the function defined by

$$(g \circ f)(x) = g(f(x)).$$

The domain of $g \circ f$ is the set of values of x in the domain of f for which $f(x)$ is in the domain of g.

The composite $g \circ f$ can be written without parentheses when applied to an argument, as in $g \circ f(x)$. Parentheses make it easier to see that there is one composite function applied to the argument, as in the following.

$$\begin{aligned}(f \circ m)(\text{John Quincy Adams}) &= f(m(\text{John Quincy Adams}))\\ &= f(\text{Abigail Smith})\\ &= \text{Rev. William Smith}\end{aligned}$$

Composition of Functions Is Not Commutative

The operation that yields the composite of two functions is called **function composition**. Order makes a difference in function composition.

$$\begin{aligned}(m \circ f)(\text{John Quincy Adams}) &= m(f(\text{John Quincy Adams}))\\ &= m(\text{John Adams Jr.})\\ &= \text{Susanna Boylston}\end{aligned}$$

The mother of the father of John Quincy Adams is Susanna Boylston. The function $m \circ f$ might be called the "paternal grandmother" function.

Notice that the two functions $f \circ m$ and $m \circ f$ are different functions. The range of $f \circ m$ contains only men, while the range of $m \circ f$ contains only women. This illustrates that *composition of functions is not commutative*. The next two examples show this with functions of real numbers.

Example 1

Let f and g be defined by $f(x) = 2x^2 + 3x$ and $g(x) = x - 7$. Evaluate.

a. $(f \circ g)(-2)$ **b. $(g \circ f)(-2)$**

Solution

a. To evaluate $(f \circ g)(-2)$, first evaluate $g(-2)$.

$$g(-2) = -2 - 7 = -9$$

Then use this output as the input to f. So

$$\begin{aligned}f(g(-2)) &= f(-9)\\ &= 2(-9)^2 + 3(-9) = 135.\end{aligned}$$

b. $g \circ f(-2) = g(f(-2)) = g(2(-2)^2 + 3(-2))$
$$= g(2)$$
$$= 2 - 7 = -5$$

It can be tedious to evaluate points for composites of functions. An alternative is to find a formula for the composite.

GUIDED

Example 2

Let $f(x) = 2x^2 + 3x$ and $g(x) = x - 7$.

a. Derive a formula for $(f \circ g)(x)$.

b. Give a simplified formula for $(g \circ f)(x)$.

c. Verify that $f \circ g \neq g \circ f$ by graphing.

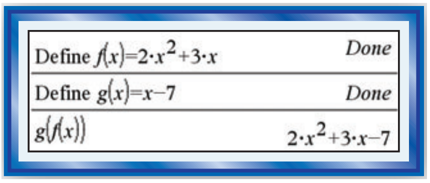
Solution

a. In $(f \circ g)(x) = f(g(x))$, first substitute $x - 7$ for $g(x)$.

$$f(g(x)) = f(x - 7)$$

Now use $x - 7$ as the input to function f.

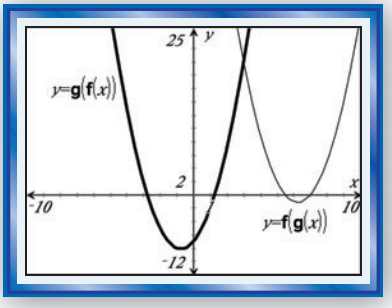
$$f(x - 7) = 2(\underline{\ ?\ })^2 + 3(\underline{\ ?\ })$$
$$= 2(\underline{\ ?\ } - \underline{\ ?\ } + 49) + 3x - 21$$
$$= \underline{\ ?\ }$$



b. To find a formula for $(g \circ f)(x)$, substitute the expression for $f(x)$ first, or use a CAS. Define the functions on your CAS and evaluate the composite directly.

c. Graph the two functions, $f \circ g$ and $g \circ f$, using a graphing utility. You can see that the graphs are different. So the functions are different.

Notice that although $f \circ g$ and $g \circ f$ are ***not*** the same function, there is at least one value of x at which they have the same y-value. This is the x-value at the point of intersection of the two parabolas.



STOP QY

Finding the Domain of a Composite Function

The domain of a composite function $g \circ f$ is the set of all elements for which $g \circ f$ is defined. So, to be in the domain of $g \circ f$, a number x must be in the domain of f, and the corresponding $f(x)$ value must be in the domain of g.

QY

True or False $f \circ g$ is the image of the graph of f under a translation 7 units to the right.

Example 3

Let f and g be real functions defined by $f(m) = \sqrt{m}$ and $g(m) = \frac{2}{m-3}$. Find the domain of $g \circ f$.

Solution 1 Because f is a real function, the domain of f is the set of all nonnegative real numbers. The domain of g is the set of all real numbers but 3, so all values of $f(m)$ except when $\sqrt{m} = 3$ are in the domain of g. Thus, the domain of $g \circ f$ is the set of real numbers m with $m \geq 0$ and $m \neq 9$.

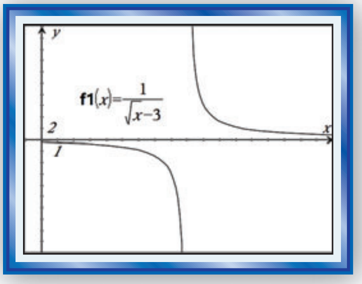
Solution 2 Find a formula for $g \circ f$ and analyze the domain.

$(g \circ f)(m) = g(f(m)) = g(\sqrt{m}) = \frac{1}{\sqrt{m}-3}$

$\sqrt{m}$ is defined in the real number system only for $m \geq 0$.

$\sqrt{m} - 3 = 0$ when $m = 9$. So, the domain of $g \circ f$ is $\{m: m \geq 0 \text{ and } m \neq 9\}$.

Check Use a graphing utility to graph $g(f(x)) = \frac{1}{\sqrt{x}-3}$. You should get a graph like the one at the right with an asymptote at $x = 9$.



Example 3 shows that the domain of the composite can be different from the domain of either of the component functions.

Composition of Transformations

Because transformations are functions, they can be composed. Like other functions, composition of transformations is not commutative.

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Example 4

Let $S:(x, y) \rightarrow (2x, y)$ and let $T:(x, y) \rightarrow (x + 4, y - 3)$.

a. Describe S and T in words.

b. Write a formula for the composite $(T \circ S)(x, y)$ and describe it in words.

c. Write a formula for the composite $(S \circ T)(x, y)$ and describe it in words.

Solution

a. S is a horizontal scale change of magnitude __?__ and T is a __?__.

b. $(T \circ S)(x, y) = T(S(x, y)) = T(2x, y) = (2x + 4, y - 3)$. $T \circ S$ is a horizontal scale change of magnitude 2, followed by a translation 4 units right and 3 units down.

c. $(S \circ T)(x, y) = S(T(x, y)) = S($__?__, __?__$) = (2($__?__$),$ __?__$) =$ (__?__, __?__). $T \circ S$ is a translation 4 units right and 3 units down, followed by a horizontal scale change of magnitude 2.

Questions

COVERING THE IDEAS

In 1 and 2, consider John Quincy Adams's family tree.

1. What biological relation does the composite $m \circ m$ represent?
2. John Quincy Adams married Louisa Catherine Johnson. Suppose they have a child, x.
   a. Evaluate $(m \circ f)(x)$. b. Explain why $(f \circ m)(x) \neq (m \circ f)(x)$.

In 3 and 4, refer to the functions *f* and *g* of Examples 1 and 2.

3. Verify that $f(g(0)) \neq g(f(0))$.
4. **True or False** a. $f(g(6)) = g(f(6))$ b. $f(g(3)) = g(f(3))$
5. Let $M(t) = 2t - 1$ and $N(t) = \frac{3}{t+1}$.
   a. Find a formula for $(M \circ N)(t)$. b. State the domain of $M \circ N$.
6. **True or False** Composition of functions is commutative.

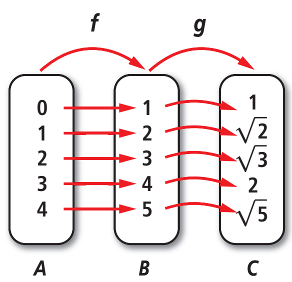
In 7 and 8, let $f(x) = (x + 1)^2$ and $g(x) = x - 2$.

7. Evaluate $f(g(-5))$ and $g(f(-5))$.
8. Show that $g \circ f \neq f \circ g$ by graphing.

In 9 and 10, $f(x) = 2x^3 - 1$ and $g(x) = 3x$.

9. Evaluate each expression.
   a. $f(g(-1))$ b. $(f \circ g)(0)$
10. a. Find a formula for $(f \circ g)(x)$.
    b. State the domain of $f \circ g$.
    c. For what value of x does $(f \circ g)(x) = (g \circ f)(x)$?
11. Let T be a transformation that translates each point right 3 and up 1; let S be a vertical scale change of magnitude $\frac{1}{4}$.
    a. **Fill in the Blanks** $T: (x, y) \rightarrow (\underline{\ ?\ }, \underline{\ ?\ })$ and $S: (x, y) \rightarrow (\underline{\ ?\ }, \underline{\ ?\ })$.
    b. Find a formulas for $T \circ S$ and $S \circ T$.

APPLYING THE MATHEMATICS



12. Consider the sets A, B, and C at the right.
    a. Evaluate $g(f(3))$.
    b. The composite $g \circ f$ maps 4 to what number?
    c. If $f(x) = x + 1$ and $g(x) = \sqrt{x}$, write a formula for $g(f(x))$.
    d. If the domain of f is extended to the set of all reals, what is the domain of $g \circ f$?
13. If $S(x, y) = (-y, x)$, find a formula for $S \circ S$.
14. If $T(x, y) = (x + 6, 2y)$, find a formula for $T \circ T$.

In 15–17, suppose $D(x) = 0.9x$ and $R(x) = x - 100$.

15. Explain why it is appropriate to call D a discount function and R a rebate function.

16. a. Evaluate $D(R(1200))$ and $R(D(1200))$.
    b. If you are buying a flat screen TV for \$1200, is it better to apply the discount after the rebate or before the rebate?

17. Find rules for $D \circ R(x)$ and for $R \circ D(x)$. Prove that $D \circ R \neq R \circ D$.

18. From these tables, evaluate each expression, if possible.

x	-5	-3	-1	0	1	2	3	4	5
$f(x)$	-5	-4	-3	0	3	2	1	0	-1

x	0	1	4	9	16	25
$g(x)$	0	1	2	3	4	5

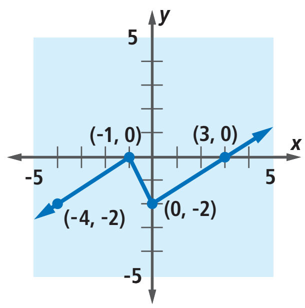
a. $f(g(4))$ b. $g(f(4))$ c. $f(g(1))$ d. $g(f(1))$

19. a. One mile is 5280 feet. Write a formula for a function m that converts number of feet to number of miles.
    b. One mile is exactly 1.609344 kilometers. Write a formula for a function k that converts number of miles to number of kilometers.
    c. Write a rule for a composite function that converts feet to kilometers.
    d. How many feet is 5 kilometers?

20. Let f and g be real functions with $f(x) = \frac{1}{x}$ and $g(x) = x + 4$.
    a. Give equations for all asymptotes of the graph of $f \circ g$.
    b. Give equations for all asymptotes of the graph of $g \circ f$.

REVIEW

21. The graph of $y = f(x)$ is drawn at the right. Draw the graph of $y = 2f(3x)$. **(Lesson 3-5)**



22. Prices of pies at Benny's Bakery have a mean of \$13.58 and a variance of 2.18. Assuming that customers do not change their purchasing pattern, what will be the effect on the mean and standard deviation under each circumstance? **(Lessons 3-6, 3-3)**
    a. The price per pie increases by 50 cents.
    b. The price per pie increases by 5%.

23. Find an equation for the image of the graph of $f(x) = x^2$ under the transformation $S: (x, y) \rightarrow \left(\frac{x}{4}, y\right)$. **(Lesson 3-5)**

24. ***Skill Sequence*** Find all real solutions. **(Previous Course)**
    a. $(5x - 10)(x - 3) = 0$ b. $3x^2 = 1 - 2x$ c. $3x^4 = 1 - 2x^2$

EXPLORATION

25. Find a function f such that $f(f(x)) = x$ yet $f(x) \neq x$.

QY ANSWER

true